

LORENTZIAN SIMILARITY MANIFOLD

YOSHINOBU KAMISHIMA

ABSTRACT. If an $m + 2$ -manifold M is locally modeled on \mathbb{R}^{m+2} with coordinate changes lying in the subgroup $G = \mathbb{R}^{m+2} \rtimes (\mathrm{O}(m+1, 1) \times \mathbb{R}^+)$ of the affine group $\mathrm{A}(m+2)$, then M is said to be a *Lorentzian similarity manifold*. A Lorentzian similarity manifold is also a conformally flat Lorentzian manifold because G is isomorphic to the stabilizer of the Lorentz group $\mathrm{PO}(m+2, 2)$ which is the full Lorentzian group of the Lorentz model $S^{2n+1,1}$. It contains a class of Lorentzian flat space forms. We shall discuss the properties of compact Lorentzian similarity manifolds using developing maps and holonomy representations.

1. INTRODUCTION

Let $\mathrm{A}(m+2) = \mathbb{R}^{m+2} \rtimes \mathrm{GL}(m+2, \mathbb{R})$ be the affine group of the $m+2$ -dimensional euclidean space \mathbb{R}^{m+2} . An $m+2$ -manifold M is an *affinely flat* manifold if M is locally modeled on \mathbb{R}^{m+2} with coordinate changes lying in $\mathrm{A}(m+2)$. When \mathbb{R}^{m+2} is endowed with a Lorentz inner product, we obtain *Lorentz similarity geometry*

$$\mathrm{Sim}_L(\mathbb{R}^{m+2}) = \mathbb{R}^{m+2} \rtimes (\mathrm{O}(m+1, 1) \times \mathbb{R}^+)$$

as a subgeometry of $\mathrm{A}(m+2)$. If an affinely flat manifold M is locally modeled on $\mathrm{Sim}_L(\mathbb{R}^{m+2})$, then M is said to be a *Lorentzian similarity manifold*. Lorentzian similarity geometry contains *Lorentzian flat geometry* $(\mathrm{E}(m+1, 1), \mathbb{R}^{m+2})$ where $\mathrm{E}(m+1, 1) = \mathbb{R}^{m+2} \rtimes \mathrm{O}(m+1, 1)$.

Theorem A. *If M is a compact complete Lorentzian similarity manifold, then M is a Lorentzian flat space form. Furthermore, M is diffeomorphic to an infrasolvmanifold.*

Theorem A is proved as follows (cf. Section 2); The fundamental group $\pi_1(M)$ of a compact complete Lorentzian similarity manifold M is shown to be virtually solvable. Then we prove that $\pi_1(M)$ admits a

Date: October 11, 2011.

2000 Mathematics Subject Classification. 53C55, 57S25, 51M10.

Key words and phrases. Lorentzian similarity structure, Conformally flat Lorentzian structure, Uniformization, Holonomy group, Developing map.

This research was initiated during the stay at ESI in July 2011. He gratefully acknowledges the support of the University of Vienna via the ESI.

nontrivial translation subgroup. Using these results, M will be a compact complete Lorentzian flat manifold. In particular, the Auslander-Milnor conjecture is true for compact complete Lorentzian similarity manifolds (cf. [18]).

Let $(\mathrm{PO}(m+2, 2), S^{m+1,1})$ be *conformally flat Lorentzian geometry*. If a point $\infty \in S^{m+1,1}$ is defined as the projectivization of a null vector in \mathbb{R}^{m+4} , the stabilizer $\mathrm{PO}(m+2, 2)_\infty$ is isomorphic to $\mathrm{Sim}_L(\mathbb{R}^{m+2})$ for which there is a suitable conformal Lorentzian embedding of \mathbb{R}^{m+2} into $S^{m+1,1} - \{\infty\}$ which is equivariant with respect to $\mathrm{Sim}_L(\mathbb{R}^{m+2}) = \mathrm{PO}(m+2, 2)_\infty$ (cf. [12]). In contrast to *conformally flat Riemannian geometry*, \mathbb{R}^{m+2} is properly contained in the complement $S^{m+1,1} - \{\infty\}$ (cf. [1]). A Lorentzian similarity geometry $(\mathrm{Sim}_L(\mathbb{R}^{m+2}), \mathbb{R}^{m+2})$ is a sort of subgeometry of conformally flat Lorentzian geometry $(\mathrm{PO}(m+2, 2), S^{m+1,1})$.

In general, the structure group of a conformally flat Lorentzian manifold belongs to $\mathrm{O}(m+1, 1) \times \mathbb{R}^+$. Let $\mathrm{Sim}^*(\mathbb{R}^m) = \mathbb{R}^m \rtimes (\mathrm{O}(m) \times \mathbb{R}^*)$ be the similarity subgroup of $\mathrm{O}(m+1, 1)$.

Take a subgroup $\mathrm{Sim}^*(\mathbb{R}^m) \times \mathbb{R}^+$ in $\mathrm{O}(m+1, 1) \times \mathbb{R}^+$. We call M a *conformally flat Lorentzian parabolic manifold* if the structure group is conjugate to a subgroup of $\mathrm{Sim}^*(\mathbb{R}^m) \times \mathbb{R}^+$. (See Definition 4.1.) In Section 4 we prove (cf. Theorem 4.5)

Theorem B. *Let M be an $m+2$ -dimensional compact conformally flat Lorentzian manifold whose holonomy group is virtually solvable in $\mathrm{Sim}_L(\mathbb{R}^{m+2})$. Then M is either a conformally flat Lorentzian parabolic manifold or finitely covered by the Lorentz model $S^1 \times S^{m+1}$, a Hopf manifold $S^{m+1} \times S^1$, or a torus T^{m+2} .*

For $m = 2n$, there is the natural embedding $\mathrm{U}(n+1, 1) \rightarrow \mathrm{O}(2n+2, 2)$ so that $(\mathrm{U}(n+1, 1), S^1 \times S^{2n+1})$ is a subgeometry of $(\mathrm{O}(2n+2, 2), S^1 \times S^{2n+1})$. Here $S^1 \times S^{2n+1}$ is a two-fold covering of $S^{2n+1,1}$. A $2n+2$ -dimensional manifold M is said to be a *conformally flat Fefferman-Lorentz parabolic manifold* if M is uniformized with respect to $(\mathrm{U}(n+1, 1), S^1 \times S^{2n+1})$. (Compare [15].) We study which compact conformally flat Fefferman-Lorentz parabolic manifolds are the quotients of domains of $S^{m+1,1} - \{\infty\}$ by properly discontinuous subgroups of $\mathrm{PO}(m+2, 2)_\infty$ in Section 6. See [14] for a related work.

Theorem C. *Let M be a $2n+2$ -dimensional compact conformally flat Fefferman-Lorentz parabolic manifold and*

$$(\rho, \mathrm{dev}) : (\pi_1(M), \tilde{M}) \rightarrow (\mathrm{U}(n+1, 1)^\sim, \mathbb{R} \times S^{2n+1})$$

the developing pair. Suppose that the holonomy group Γ is discrete in $\mathrm{U}(n+1, 1)^\sim$. If the developing map $\mathrm{dev} : \tilde{M} \rightarrow \mathbb{R} \times S^{2n+1,1}$ misses a closed subset which is invariant under \mathbb{R} and Γ , then dev is a covering map onto its image.

For noncompact complete Lorentzian case, i.e., properly discontinuous actions of free groups on complete simply connected Lorentzian flat manifolds, see [4], [10], [1] for details.

2. LORENTZIAN SIMILARITY MANIFOLD

Consider the following exact sequence:

$$(2.1) \quad 1 \rightarrow \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \rightarrow \text{Sim}_L(\mathbb{R}^{m+2}) \xrightarrow{P} \text{O}(m+1, 1) \rightarrow 1.$$

Lemma 2.1. *Let $M = \mathbb{R}^{m+2}/\Gamma$ be a compact complete Lorentzian similarity manifold where $\Gamma \leq \text{Sim}_L(\mathbb{R}^{m+2})$. Suppose that $P(\Gamma)$ is discrete in $\text{O}(m+1, 1)$. If $\Delta = (\mathbb{R}^{m+2} \rtimes \mathbb{R}^+) \cap \Gamma$, then $\Delta \leq \mathbb{R}^{m+2}$ which is nontrivial.*

Proof. Since $P(\Gamma)$ is discrete, it acts properly discontinuously on the $m+1$ -dimensional hyperbolic space $\mathbb{H}_{\mathbb{R}}^{m+1} = \text{O}(m+1) \times \text{O}(1) \backslash \text{O}(m+1, 1)$. The (virtually) cohomological dimension vcd of $P(\Gamma)$ satisfies $\text{vcd}(P(\Gamma)) \leq m+1$. On the other hand, the cohomological dimension $\text{cd}(\Gamma) = m+2$, the intersection Δ of (2.1) is nontrivial. Let

$$1 \rightarrow \mathbb{R}^{m+2} \rightarrow \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \xrightarrow{p} \mathbb{R}^+ \rightarrow 1$$

be the exact sequence. If $p(\Delta)$ is nontrivial, then we may assume that there exists an element $\gamma = (a, \lambda) \in \Delta$ such that $p(\gamma) = \lambda < 1$. A calculation shows $\gamma^n = (\frac{1-\lambda^n}{1-\lambda}a, \lambda^n)$ ($\forall n \in \mathbb{Z}$). The sequence of the orbits $\{\gamma^n \cdot 0; n \in \mathbb{Z}\}$ at the origin $0 \in \mathbb{R}^{m+2}$ converges when $n \rightarrow \infty$,

$$\gamma^n \cdot 0 = \frac{1-\lambda^n}{1-\lambda}a + \lambda^n \cdot 0 = \frac{1-\lambda^n}{1-\lambda}a \rightarrow \frac{1}{1-\lambda}a.$$

As Δ acts properly discontinuously on \mathbb{R}^{m+2} , $\{\gamma^n; n = 1, 2, \dots\}$ is a finite set. Since Δ is torsionfree, $\gamma = 1$ which is a contradiction. So $p(\Gamma)$ must be trivial. \square

Proposition 2.2. *Let $M = \mathbb{R}^{m+2}/\Gamma$ be a compact complete Lorentzian similarity manifold. Then Γ is virtually solvable in $\text{Sim}_L(\mathbb{R}^{m+2})$.*

Proof. (1) When $P(\Gamma)$ is discrete, we obtain the following exact sequences from (2.1).

$$(2.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{R}^{m+2} & \longrightarrow & \text{Sim}_L(\mathbb{R}^{m+2}) & \xrightarrow{L} & \text{O}(m+1, 1) \times \mathbb{R}^+ \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \xrightarrow{L} & L(\Gamma) \longrightarrow 1 \end{array}$$

If $\Delta \cong \mathbb{Z}^k$, then the span \mathbb{R}^k of Δ in \mathbb{R}^{m+2} is normalized by Γ . Let \langle, \rangle be the Lorentz inner product on \mathbb{R}^{m+2} . The rest of the argument

is similar to that of [11]. In fact, $L(\Gamma)$ of (2.2) induces a properly discontinuous affine action ρ on \mathbb{R}^{m+2-k} with finite kernel $\text{Ker } \rho$:

$$\rho : L(\Gamma) \rightarrow \text{Aff}(\mathbb{R}^{m+2-k}).$$

(Compare Lemma 3.1.) If necessary, we can find a torsionfree normal subgroup of finite index in $\rho(L(\Gamma))$ by Selberg's lemma. Passing to a finite index subgroup if necessary, the quotient $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a compact complete affinely flat manifold.

Suppose that $\langle \cdot, \cdot \rangle|_{\mathbb{R}^k}$ is nondegenerate. According to whether $\langle \cdot, \cdot \rangle|_{\mathbb{R}^k}$ is positive definite or indefinite, $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a compact complete Lorentzian similarity manifold or Riemannian similarity manifold respectively.

If $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a Lorentzian similarity manifold, by induction hypothesis, $L(\Gamma)$ is virtually solvable. When $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a Riemannian similarity manifold, i.e. $\rho(L(\Gamma)) \leq \text{Sim}(\mathbb{R}^{m+2-k})$ which is an amenable Lie group, a discrete subgroup $\rho(L(\Gamma))$ is virtually solvable by Tits' theorem. (Compare [18]. Furthermore, $\mathbb{R}^{m+2-k}/\rho(L(\Gamma))$ is a Riemannian flat manifold by Fried's theorem [7].) In each case, Γ is virtually solvable.

If $\langle \cdot, \cdot \rangle|_{\mathbb{R}^k}$ is degenerate, then $\mathbb{R}^k = \mathbf{R}$ consisting of a lightlike vector as a basis. The holonomy group $L(\Gamma)$ leaves invariant \mathbf{R} . The subgroup of $\text{O}(m+1, 1) \times \mathbb{R}^+$ preserving \mathbf{R} is isomorphic to $\text{Sim}^*(\mathbb{R}^m) \times \mathbb{R}^+ = (\mathbb{R}^m \rtimes (\text{O}(m) \times \mathbb{R}^*)) \times \mathbb{R}^+$ which is an amenable Lie group. As $L(\Gamma) \leq \text{Sim}^*(\mathbb{R}^m) \times \mathbb{R}^+$, $L(\Gamma)$ is virtually solvable so is Γ .

(2) When $P(\Gamma)$ is indiscrete, it follows from [20, Theorem 8.24] that the identity component of the closure $\overline{P(\Gamma)}^0$ is solvable in $\text{O}(m+1, 1)$. It belongs to the maximal amenable subgroup up to conjugate:

$$\overline{P(\Gamma)}^0 \leq \mathbb{R}^m \rtimes (\text{O}(m) \times \mathbb{R}^*).$$

It is easy to check that the normalizer of $\overline{P(\Gamma)}^0$ is still contained in $\mathbb{R}^m \rtimes (\text{O}(m) \times \mathbb{R}^*)$ because the normalizer leaves invariant at most two points $\{0, \infty\}$ on the boundary $S^m = \partial \mathbb{H}_{\mathbb{R}}^{m+1}$ for which $\text{O}(m+1, 1)_{\infty} = \mathbb{R}^m \rtimes (\text{O}(m) \times \mathbb{R}^*)$. Hence $P(\Gamma) \leq \mathbb{R}^m \rtimes (\text{O}(m) \times \mathbb{R}^*)$. There is an exact sequence induced from (2.1):

$$1 \rightarrow \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \rightarrow P^{-1}(\mathbb{R}^m \rtimes (\text{O}(m) \times \mathbb{R}^*)) \xrightarrow{P} \mathbb{R}^m \rtimes (\text{O}(m) \times \mathbb{R}^*) \rightarrow 1$$

in which $P^{-1}(\mathbb{R}^m \rtimes (\text{O}(m) \times \mathbb{R}^*))$ is an amenable Lie subgroup. Hence, Γ is virtually solvable. □

Proposition 2.3. *Let M be a compact complete Lorentzian similarity manifold \mathbb{R}^{m+2}/Γ . Then M is diffeomorphic to an infrasolvmanifold U/Γ .*

Proof. As $\Gamma \leq \mathbb{R}^{m+2} \rtimes (\mathrm{O}(m+1, 1) \times \mathbb{R}^+)$ is a virtually solvable group, take the real algebraic hull $A(\Gamma) = U \cdot T$ where U is a unipotent radical and T is a reductive d -subgroup such that T/T^0 is finite. Then each element $r = u \cdot t \in U \cdot T$ acts on U by $\gamma x = utxt^{-1}$ ($x \in U$). It follows from the result of [2] that Γ acts properly discontinuously on U such that U/Γ is compact. Furthermore U/Γ is diffeomorphic to an infrasolvmanifold by [2, Theorem 1.2].

Since U/Γ is compact, we choose a compact subset $D \subset U$ such that $U = \Gamma \cdot D$. As Γ acts properly discontinuously on \mathbb{R}^{m+2} and $U \cdot T \leq \mathbb{R}^{m+2} \rtimes (\mathrm{O}(m+1, 1) \times \mathbb{R}^+)$, it is easily checked that U acts properly on \mathbb{R}^{m+2} . Since T is reductive, we may assume that $T \cdot 0 = 0 \in \mathbb{R}^{m+2}$. Define a map;

$$\rho : U \rightarrow \mathbb{R}^{m+2}, \quad \rho(x) = x \cdot 0.$$

Noting that U acts freely on \mathbb{R}^{m+2} , ρ is a simply transitive action. For $\gamma = u \cdot t \in \Gamma$, $\gamma x = utxt^{-1}$ as above. Then $\rho(\gamma x) = utxt^{-1} \cdot 0 = utx \cdot 0 = \gamma \rho(x)$. So ρ is Γ -equivariant, ρ induces a diffeomorphism on the quotients; $U/\Gamma \cong \mathbb{R}^{m+2}/\Gamma$. □

Proposition 2.4. *The fundamental group Γ of a compact complete Lorentzian similarity manifold \mathbb{R}^{m+2}/Γ admits a nontrivial translation subgroup. In particular, the fundamental group of a compact Lorentzian flat space form admits a nontrivial translation subgroup.*

Proof. Let Γ_0 be a finite index solvable subgroup of Γ and $A(\Gamma_0) = U \cdot T$ the real algebraic hull for Γ_0 as above. Let $L : \Gamma_0 \rightarrow L(\Gamma_0)$ be the holonomy homomorphism as in (2.2). As the real algebraic hull for $L(\Gamma_0)$ can be taken inside $\mathrm{O}(m+1, 1) \times \mathbb{R}^+$, L extends naturally to a homomorphism $L : A(\Gamma_0) \rightarrow A(L(\Gamma_0))$. We have the following commutative diagram:

$$(2.3) \quad \begin{array}{ccccc} \mathbb{R}^{m+2} & \longrightarrow & \mathrm{Sim}_L(\mathbb{R}^{m+2}) & \xrightarrow{L} & \mathrm{O}(m+1, 1) \times \mathbb{R}^+ \\ & & \uparrow & & \uparrow \\ & & A(\Gamma_0) & \xrightarrow{L} & A(L(\Gamma_0)) \\ & & \uparrow & & \uparrow \\ \Gamma_0 & & & \xrightarrow{L} & L(\Gamma_0). \end{array}$$

Suppose that $\mathbb{R}^{m+2} \cap \Gamma_0 = \{1\}$ so that $L : \Gamma_0 \rightarrow L(\Gamma_0)$ is isomorphic. Then $L : A(\Gamma_0) \rightarrow A(L(\Gamma_0))$ is also isomorphic (cf. [2]). Since $A(\Gamma_0) =$

$U \cdot T$, this implies $A(L(\Gamma_0)) = L(U) \cdot L(T)$. If we note that $A(L(\Gamma_0))$ is a solvable real linear algebraic group in $O(m+1, 1) \times \mathbb{R}^+$, it follows

$$(2.4) \quad A(L(\Gamma_0)) \leq (\mathbb{R}^m \rtimes (T^k \times \mathbb{R}^*)) \times \mathbb{R}^+.$$

Here T^k is a maximal torus in $O(m)$ for which $T^k \times \mathbb{R}^*$ acts on \mathbb{R}^m as similarities. As $L(U)$ is a connected simply connected unipotent Lie group, it follows $L(U) \leq \mathbb{R}^m \times \mathbb{R}^+$. Thus, $\dim L(U) \leq m+1$. On the other hand, U/Γ_0 is an $m+2$ -dimensional compact aspherical manifold, we note that $\text{Rank } \Gamma_0 = \dim U = m+2$. This contradicts that $L : \Gamma_0 \rightarrow L(\Gamma_0)$ is isomorphic. Therefore $\mathbb{R}^{m+2} \cap \Gamma_0 \leq \mathbb{R}^{m+2} \cap \Gamma$ is nontrivial. \square

Proposition 2.5. *Every compact complete Lorentzian similarity manifold is a Lorentzian flat space form.*

Proof. Consider the exact sequences:

$$(2.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & E(m+1, 1) & \longrightarrow & \text{Sim}_L(\mathbb{R}^{m+2}) & \xrightarrow{q} & \mathbb{R}^+ \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma & \xrightarrow{q} & q(\Gamma) \longrightarrow 1 \end{array}$$

where $\Gamma_1 = E(m+1, 1) \cap \Gamma$. It is enough to show that $q(\Gamma)$ is trivial. Suppose that there exists an element $\gamma = (a, \lambda A) \in \Gamma$ such that

$$q(\gamma) = \lambda < 1.$$

By Proposition 2.4, let $\mathbb{R}^{m+2} \cap \Gamma \cong \mathbb{Z}^\ell$ for some $\ell \geq 1$.

Let $\langle \cdot, \cdot \rangle$ be the Lorentz inner product on \mathbb{R}^{m+2} as before.

(1) Suppose $\ell \geq 1$. Then there exists a vector $n \in \mathbb{Z}^k$ such that $\langle n, n \rangle \neq 0$. Calculate

$$\gamma n \gamma^{-1} = (a, \lambda A)(n, I)(-A^{-1}a, \lambda^{-1}A^{-1}) = (\lambda A n, I)$$

so that $\gamma^k n \gamma^{-k} = (\lambda^k A^k n, I)$. Take a sequence of orbits at the origin $\{\gamma^k n \gamma^{-k} \cdot 0; k = 0, 1, 2, \dots\}$ in \mathbb{R}^{m+2} . As $\gamma^k n \gamma^{-k} \cdot 0 = \lambda^k A^k n$, it follows

$$\langle \lambda^k A^k n, \lambda^k A^k n \rangle = \lambda^{2k} \langle A^k n, A^k n \rangle = \lambda^{2k} \langle n, n \rangle \rightarrow 0 \quad (k \rightarrow \infty).$$

Noting $\langle n, n \rangle \neq 0$, this implies that $\gamma^k n \gamma^{-k} \cdot 0 \rightarrow 0$ ($k \rightarrow \infty$). As Γ acts properly discontinuously, $\{\gamma^k n \gamma^{-k}\}$ is a finite set, i.e. $\gamma^k n \gamma^{-k} = 1$ for some k . Thus $n = 1$ which is a contradiction.

(2) Suppose $\mathbb{R}^{m+2} \cap \Gamma \cong \mathbb{Z}$ which is generated by a null vector n , i.e. $\langle n, n \rangle = 0$. Since Γ leaves \mathbb{Z} invariant, taking a subgroup of index 2 (if necessary), we may assume $n = \gamma n \gamma^{-1} = (\lambda A n, I)$ for $\gamma = (a, \lambda A) \in \Gamma$. This implies $A n = \lambda^{-1} n$.

Let $\{\ell_1, e_2, \dots, e_{m+1}, \ell_{m+2}\}$ be the basis on \mathbb{R}^{m+2} such that

$$\langle \ell_1, \ell_1 \rangle = \langle \ell_{m+2}, \ell_{m+2} \rangle = 0, \langle e_i, e_j \rangle = \delta_{ij}, \langle \ell_1, \ell_{m+2} \rangle = 1.$$

The subgroup $\text{Sim}(\mathbb{R}^m)$ of $O(m+1, 1)$ has the form with respect to the above basis:

$$(2.6) \quad \text{Sim}(\mathbb{R}^m) = \left\{ A = \begin{pmatrix} \lambda^{-1} & x & -\frac{\lambda|x|^2}{2} \\ 0 & B & -\lambda B^t x \\ 0 & 0 & \lambda \end{pmatrix} \mid \begin{array}{l} \lambda \in \mathbb{R}^+, \\ B \in O(m), \\ x \in \mathbb{R}^m. \end{array} \right\}.$$

See [15] for details. We may take n for the null vector ℓ_1 . Since $An = \lambda^{-1}n$, A has the form as in (2.6). Then we can write

$$(2.7) \quad \gamma = (a, \lambda A) = \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{pmatrix} 1 & \lambda x & -\lambda^2|x|^2/2 \\ 0 & \lambda B & -\lambda^2 B^t x \\ 0 & 0 & \lambda^2 \end{pmatrix} \right)$$

where $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}^{m+1}$. If we put $\rho(\gamma) = (a_2, \begin{pmatrix} \lambda B & -\lambda^2 B^t x \\ 0 & \lambda^2 \end{pmatrix}) \in A(m+1)$, then the matrix $\begin{pmatrix} \lambda B & -\lambda^2 B^t x \\ 0 & \lambda^2 \end{pmatrix}$ has no eigenvalue 1 so that $\rho(\gamma)$ has a fixed point $y \in \mathbb{R}^{m+1}$, i.e. $\rho(\gamma)(y) = y$. Conjugate Γ by a translation $t_y = (\begin{bmatrix} 0 \\ -y \end{bmatrix}, I)$, it follows

$$(2.8) \quad t_y \gamma t_y^{-1} = \left(\begin{bmatrix} c \\ 0 \end{bmatrix}, \begin{pmatrix} 1 & \lambda x & -\lambda^2|x|^2/2 \\ 0 & \lambda B & -\lambda^2 B^t x \\ 0 & 0 & \lambda^2 \end{pmatrix} \right)$$

where $c = a_1 + (\lambda x, -\lambda^2|x|^2/2) \cdot y \in \mathbb{R}$.

When we consider the orbits of $\{t_y \gamma^k t_y^{-1}\}$ at the origin $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{m+2}$ ($k = 1, 2, \dots$), it follows

$$(2.9) \quad t_y \gamma^k t_y^{-1} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} kc \\ 0 \end{pmatrix}.$$

On the other hand, noting $t_y n t_y^{-1} = n$, we put $n = (\begin{bmatrix} t \\ 0 \end{bmatrix}, I)$.

Case I. $\frac{c}{t}$ is rational, say $\frac{p}{q}$. Take the element $t_y \gamma^q t_y^{-1} \cdot n^{-p} \in t_y \Gamma t_y^{-1}$. Then it follows

$$(2.10) \quad t_y \gamma^q t_y^{-1} \cdot n^{-p} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} qc - pt \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $t_y \Gamma t_y^{-1}$ acts freely on \mathbb{R}^{m+2} , this shows $t_y \gamma^q t_y^{-1} \cdot n^{-p} = 1$, and thus $\gamma^q = t_y^{-1} n^p t_y = n^p$. The linear part of γ^q is $(\lambda A)^q$ for $\gamma = (a, \lambda A)$, so it follows $(\lambda A)^q = I$. By the formula of (2.7), we obtain $\lambda^{2q} = 1$. This is impossible because $\lambda < 1$ for the element γ .

Case II. $\frac{c}{t}$ is irrational. Let $\lim_{i \rightarrow \infty} \frac{\ell_i}{m_i} = \frac{c}{t}$, equivalently there exist integers m_i, ℓ_i such that $m_i c - \ell_i t \rightarrow 0$ ($i \rightarrow \infty$). Take a sequence of elements $\{t_y \gamma^{m_i} t_y^{-1} \cdot n^{-\ell_i} \mid i = 1, 2, \dots\}$ in $t_y \Gamma t_y^{-1}$ and evaluate at the origin:

$$(2.11) \quad t_y \gamma^{m_i} t_y^{-1} \cdot n^{-\ell_i} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_i c - \ell_i t \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (i \rightarrow \infty).$$

By properness of $t_y \Gamma t_y^{-1}$, $\{t_y \gamma^{m_i} t_y^{-1} \cdot n^{-\ell_i}\}$ is a finite set, say $t_y \gamma^{m_i} t_y^{-1} \cdot n^{-\ell_i} = t_y \gamma^{m_j} t_y^{-1} \cdot n^{-\ell_j}$ for some i, j . As t_y and n commute, it follows

$$\gamma^m = n^\ell \quad (\exists m, \ell \in \mathbb{Z}).$$

Again the formula of (2.7) implies $\lambda^{2m} = 1$ which is impossible for $\gamma = (a, \lambda A)$.

As a consequence, $q(\Gamma) = \{1\}$ in (2.5). □

3. LORENTZIAN FLAT SEIFERT MANIFOLDS

Let $M = \mathbb{R}^{m+2}/\Gamma$ be a compact complete Lorentzian similarity manifold. It follows from Proposition 2.4 that $\Gamma \cap \mathbb{R}^{m+2}$ is nontrivial, say \mathbb{Z}^k . Then Γ normalizes its span \mathbb{R}^k of \mathbb{Z}^k in \mathbb{R}^{m+2} . As \mathbb{R}^k acts properly on \mathbb{R}^{m+2} as translations, we have an equivariant principal bundle:

$$(3.1) \quad (\mathbb{Z}^k, \mathbb{R}^k) \longrightarrow (\Gamma, \mathbb{R}^{m+2}) \xrightarrow{\nu} (Q, \mathbb{R}^\ell)$$

where $\ell = m + 2 - k$ and $Q = \Gamma/\mathbb{Z}^k$. In this case each element γ of Γ has the form:

$$(3.2) \quad \gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right)$$

where

$$(3.3) \quad \nu(\gamma) = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad A \in \text{GL}(k, \mathbb{Z}), B \in \text{GL}(\ell, \mathbb{R}).$$

If we put

$$(3.4) \quad \rho(\nu(\gamma)) = (b, B) \in A(\ell),$$

then it is easy to see that $\rho : Q \rightarrow A(\ell)$ is a well-defined homomorphism. The quotient group Q acts on \mathbb{R}^ℓ through ρ :

$$\alpha \cdot w = \rho(\nu(\gamma))w \quad (\nu(\gamma) = \alpha \in Q, w \in \mathbb{R}^\ell.)$$

Recall the following lemma (cf. [11]).

Lemma 3.1. *The group $\rho(Q)$ is a properly discontinuous affine action on \mathbb{R}^ℓ such that*

- *Ker ρ is a finite subgroup.*
- *$\mathbb{R}^\ell / \rho(Q)$ is a compact affine orbifold.*

Proof. We show that Q acts properly discontinuously. Consider the pushout:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \Gamma & \xrightarrow{\nu} & Q \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{R}^k & \longrightarrow & \mathbb{R} \cdot \Gamma & \xrightarrow{\nu} & Q \longrightarrow 1. \end{array}$$

As both \mathbb{R}^k and Γ act freely and properly on \mathbb{R}^{m+2} with $\mathbb{R}^k / \mathbb{Z}^k = T^k$, it follows that $\mathbb{R}^k \cdot \Gamma$ acts properly on \mathbb{R}^{m+2} . Since $\mathbb{R}^k \rightarrow \mathbb{R}^{m+2} \xrightarrow{\nu} \mathbb{R}^\ell$ is a principal bundle, choose a *continuous* section $s : \mathbb{R}^\ell \rightarrow \mathbb{R}^{m+2}$ of ν . Let $\{\alpha_i\}_{i \in \mathbb{N}}$ be a sequence of Q such that

$$\alpha_i \cdot w_i \rightarrow z, \quad w_i \rightarrow w \quad (i \rightarrow \infty).$$

Choose a sequence $\{\gamma_i\}_{i \in \mathbb{N}}$ from Γ such that $\nu(\gamma_i) = \alpha_i$. As

$$\nu(\gamma_i s(w_i)) = \alpha_i \cdot w_i = \nu(s(\alpha_i w_i)),$$

there is a sequence $\{t_i\}_{i \in \mathbb{N}} \leq \mathbb{R}^k$ such that

$$t_i \gamma_i s(w_i) = s(\alpha_i w_i), \quad s(\alpha_i \cdot w_i) \rightarrow s(z), \quad s(w_i) \rightarrow s(w).$$

Since $\mathbb{R}^k \cdot \Gamma$ acts properly on \mathbb{R}^{m+2} , there is an element $g \in \mathbb{R}^k \cdot \Gamma$ such that $t_i \gamma_i \rightarrow g$ and so $\alpha_i = \nu(t_i \gamma_i) \rightarrow \nu(g) \in \Gamma$. Thus Q acts properly discontinuously on \mathbb{R}^ℓ .

We check that $\text{Ker } \rho$ is finite. Let $1 \rightarrow \mathbb{Z}^k \rightarrow \Gamma_1 \rightarrow \text{Ker } \rho \rightarrow 1$ be the induced extension by the inclusion $\text{Ker } \rho \leq Q$. Then Γ_1 acts invariantly in the inverse image $\mathbb{R}^k = \nu^{-1}(pt)$. As Γ acts freely and properly, the quotient \mathbb{R}^k / Γ_1 is a closed submanifold in M . Since $\mathbb{R}^k / \mathbb{Z}^k = T^k$ covers \mathbb{R}^k / Γ_1 , $\text{Ker } \rho$ is finite. □

By the definition [17], we obtain

Proposition 3.2. *$T^k \rightarrow M \rightarrow \mathbb{R}^\ell / \rho(Q)$ is an injective Seifert fiber space with typical fiber a torus T^k and exceptional fiber a euclidean space form T^k / F .*

In [8] Fried has found all simply transitive Lie group actions on 4-dimensional Lorentzian flat space \mathbb{R}^4 which applied to classify 4-dimensional compact (complete) Lorentzian flat manifolds M up to a finite cover. As a consequence, M is finitely covered by a solvmanifold.

We take a different approach to determine 4-dimensional compact complete Lorentzian flat manifolds M from the existence of *causal actions*.

Definition 3.3. *A circle S^1 (respectively \mathbb{R}) is a causal action on M if the vector field induced by S^1 (respectively \mathbb{R}) is causal (timelike, spacelike or lightlike) vector field on M . Compare [13].*

We have the following result which occurs particularly in dimension 4 but not in general.

Proposition 3.4. *The fundamental group Γ of a compact complete Lorentzian flat manifold M has a finite index subgroup which contains a central translation subgroup. In particular, some finite cover of M admits a causal circle action.*

Proof. Let $\mathbb{Z}^k = \Gamma \cap \mathbb{R}^4$ which is a nontrivial translation subgroup by Proposition 2.4. If $k = 1$, then \mathbb{Z} is central in a subgroup of finite index in Γ .

Case 1. Suppose that $\mathbb{Z}^2 = \Gamma \cap \mathbb{R}^4$ (which is maximal). Let

$$G = \mathbb{R}^4 \rtimes (\mathbb{R}^2 \rtimes (\mathrm{SO}(2) \times \mathbb{R}^+))$$

be the maximal connected solvable Lie subgroup of $E(3,1)$. (See the proof of (2) of Proposition 2.2.) Then Γ lies in the following exact sequences up to finite index:

$$(3.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & E(3,1) & \xrightarrow{L} & \mathrm{O}(3,1) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Gamma & \xrightarrow{L} & L(\Gamma) & \longrightarrow & 1 \\ & & \mu_P \downarrow & & \mu_P \downarrow & & \mu_P \downarrow & & \\ 1 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & G & \xrightarrow{L} & \mathbb{R}^2 \rtimes (\mathrm{SO}(2) \times \mathbb{R}^+) & \longrightarrow & 1 \end{array}$$

Here μ_P is the conjugate homomorphism by some matrix $P \in \mathrm{GL}(4, \mathbb{R})$. For $\gamma \in \Gamma$, we write

$$(3.6) \quad \gamma = \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right)$$

so that $L(\gamma) = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. The conjugation homomorphism $\phi : L(\Gamma) \rightarrow \mathrm{Aut}(\mathbb{Z}^2)$ is given by

$$\phi(L(\gamma)) = A \in \mathrm{GL}(2, \mathbb{Z}).$$

As $L(\Gamma)$ is a free abelian group of rank 2, $\phi(L(\Gamma))$ belongs to A or N up to conjugacy where $\mathrm{SL}(2, \mathbb{R}) = KAN$. Since $\mathrm{GL}(2, \mathbb{Z})$ is discrete, $\phi(L(\Gamma))$ is isomorphic to \mathbb{Z} , and so $\mathrm{Ker} \phi = \mathbb{Z}$. Choose a generator γ_0 from $\mathrm{Ker} \phi$ and $\gamma \in \Gamma$ for which $\phi(L(\gamma))$ generates $\phi(L(\Gamma))$. Note γ_0, γ and \mathbb{Z}^2 generate Γ .

Recall the homomorphism $\rho : L(\Gamma) \rightarrow \mathrm{A}(2)$ defined by $\rho(L(\gamma)) = (a_2, B)$ from (3.4). Since $\rho(L(\Gamma))$ is a properly discontinuous action of $\mathrm{A}(2)$ with compact quotient, the holonomy group of $\rho(L(\Gamma))$ is a *unipotent subgroup* of $\mathrm{GL}(2, \mathbb{R})$. In particular, each B has two eigenvalues 1 and so $L(\gamma)$ has at least two eigenvalues 1. From (3.5), $\mu_P(L(\Gamma)) \leq \mathbb{R}^2 \rtimes (\mathrm{SO}(2) \times \mathbb{R}^+)$ for which

$$(3.7) \quad \mu_P(L(\gamma)) = PL(\gamma)P^{-1} = \begin{pmatrix} \lambda^{-1} & x & -\lambda|x|^2/2 \\ 0 & T & -\lambda T^t x \\ 0 & 0 & \lambda \end{pmatrix}$$

where $T \in \mathrm{SO}(2)$. Since $L(\Gamma)$ is a free abelian group of rank 2, it follows either $\mu_P(L(\Gamma)) \leq \mathbb{R}^2$ or $\mu_P(L(\Gamma)) \leq \mathrm{SO}(2) \times \mathbb{R}^+$.

If $\mu_P(L(\Gamma)) \leq \mathrm{SO}(2) \times \mathbb{R}^+$, applying $\gamma_0 \in \mathrm{Ker} \phi$,

$$(3.8) \quad PL(\gamma_0)P^{-1} = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

As $\phi(L(\gamma_0)) = A = I$ in this case, $L(\gamma_0)$ has all eigenvalues 1. (3.8) shows $\lambda = 1$, $T = I$. Hence $PL(\gamma_0)P^{-1} = I$ or $L(\gamma_0) = I$. So $\gamma_0 \in \Gamma \cap \mathbb{R}^4$ which contradicts a maximality of the translation subgroup \mathbb{Z}^2 . It then follows $\mu_P(L(\Gamma)) \leq \mathbb{R}^2$. In this case

$$(3.9) \quad PL(\gamma)P^{-1} = \begin{pmatrix} 1 & x & -|x|^2/2 \\ 0 & I & -^t x \\ 0 & 0 & 1 \end{pmatrix}.$$

Then A of (3.6) has two eigenvalues 1 so $[\gamma, \mathbb{Z}^2] = (A - I)\mathbb{Z}^2$ has rank less than 2. Hence there is an element $m \in \mathbb{Z}^2$ such that $[\gamma, m] = 1$. As $\phi(\gamma_0) = 1$, $\gamma_0 m \gamma_0^{-1} = m$. Hence m is a central element of $\Gamma \cap \mathbb{R}^4$.

Case 2. Suppose that $\mathbb{Z}^3 = \Gamma \cap \mathbb{R}^4$. There is an induced affine action $\rho : L(\Gamma) \rightarrow \mathrm{A}(1)$ in this case so that $\rho(L(\Gamma))$ consists of a translation group up to finite index. As above we obtain

$$(3.10) \quad \gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix} \right)$$

where $A \in \mathrm{GL}(3, \mathbb{Z})$. Since $L(\gamma)$ has the eigenvalue 1, in view of (3.7), it follows either $T = I$ or $\lambda = 1$. If $T = I$, A has at least one eigenvalue 1. As $\Gamma = \mathbb{Z}^3 \rtimes \mathbb{Z}$, it follows $\mathrm{Rank}[\gamma, \mathbb{Z}^3] < 3$. Again there exists an

element $n \in \mathbb{Z}^3$ such that $\gamma n \gamma^{-1} = n$. Hence n is a central element in Γ . □

Let \mathbb{Z} be a central translation subgroup of Γ . Put $Q = \Gamma/\mathbb{Z}$. As every element $\gamma \in \Gamma$ has the form

$$(3.11) \quad \gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} 1 & C \\ 0 & B \end{pmatrix} \right)$$

where $B \in \text{GL}(3, \mathbb{R})$, there is an induced action

$$\varphi : Q \rightarrow \text{A}(3), \quad \varphi(\bar{\gamma}) = (b, B).$$

Although \mathbb{Z} is not necessarily equal to $\Gamma \cap \mathbb{R}^4$, it can be easily checked that $\varphi : Q \rightarrow \text{A}(3)$ is a properly discontinuous action such that $\mathbb{R}^3/\varphi(Q)$ is compact and $\text{Ker } \varphi$ is finite as in Lemma 3.1. If \mathbb{R} is the span of \mathbb{Z} in \mathbb{R}^4 , then \mathbb{R} is causal on \mathbb{R}^4 .

Proposition 3.5. *Every compact complete Lotentzian flat 4-manifold admits a causal circle bundle M in its finite cover.*

- (i) S^1 is a timelike circle. $M = T^4 = S^1 \times T^3$ where T^3 is a Riemannian flat torus.
- (ii) S^1 is a spacelike circle. (1) $M = S^1 \times T^3$, (2) $M = S^1 \times \mathcal{N}^3/\Delta$.
(3) $M = S^1 \times \mathcal{R}/\pi$. Each 3-dimensional factor is a Lorentzian flat manifold.
- (iii) S^1 is a lightlike circle. $M = S^1 \times \mathcal{N}^3/\Delta$ where $S^1 \rightarrow M \rightarrow S^1 \times T^2$ is a nontrivial principal bundle over the affine torus with euler number $k \in \mathbb{Z}$. Moreover, S^1 is spacelike so M coincides with (2) of case (ii).

Proof. According to whether \mathbb{R} is timelike or spacelike, we see that the induced action is Euclidean $\varphi : Q \rightarrow \text{E}(3)$ or Lorentzian $\varphi : Q \rightarrow \text{E}(2, 1)$ respectively. Moreover, we have a decomposition $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ with respect to the Lorentz inner product. Then the formula of (3.6) becomes:

$$(3.12) \quad \gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \right).$$

For $\varphi(Q) \leq \text{E}(3)$, it follows $\varphi(Q) \leq \mathbb{R}^3$ up to finite index by the Bieberbach Theorem and hence $\gamma = \left(\begin{bmatrix} a \\ b \end{bmatrix}, I \right)$. As a consequence, $\Gamma \leq \mathbb{R}^4$. This shows (i).

For $\varphi(Q) \leq \text{E}(2, 1)$, we assume $\varphi(Q)$ is torsionfree. It is known that a compact Lorentzian flat 3-manifold $\mathbb{R}^3/\varphi(Q)$ is T^3 , a Heisenberg nilmanifold \mathcal{N}/Δ or a solvmanifold \mathcal{R}/π . (For example, [9],[15].) When

$\mathbb{R}^3/\varphi(Q) = \mathcal{N}/\Delta$, the center \mathbb{R} of \mathcal{N} is the translation subgroup consisting of $\langle \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} \rangle$. The corresponding subgroup Δ in Γ belongs to

the translation subgroup $\langle \left(\begin{bmatrix} a \\ b_1 \\ 0 \\ 0 \end{bmatrix}, I \right) \rangle$. It is easy to see that Δ is a

central subgroup of rank 2.

On the other hand, there are two isomorphism classes of 4-dimensional (compact) nilmanifolds which are Nil^4/Γ or $S^1 \times \mathcal{N}/\Delta$. They are characterized as whether the center $C(\text{Nil}^4) = \mathbb{R}$ or $C(\mathbb{R} \times \mathcal{N}) = \mathbb{R}^2$. (See [21] for the classification of 4-dimensional Riemannian geometric manifolds in the sense of Thurston, Kulkarni.) By this classification, $\mathbb{R}^4/\Gamma = S^1 \times \mathcal{N}/\Delta$.

When $\mathbb{R}^3/\varphi(Q) = \mathcal{R}/\pi$, it follows that $[\pi, \pi] = \mathbb{Z}^2$. As $\mathbb{Z} \leq \Gamma$ is central, it implies $[\Gamma, \Gamma] = \mathbb{Z}^2$. By the classification [21] of 4-dimensional solvmanifolds, the universal covering group G is either one of solvable Lie groups of Inoue type $\text{Sol}_1^4 = \mathcal{N} \rtimes \mathbb{R}$, $\text{Sol}_0^4 = \mathbb{R}^3 \rtimes \mathbb{R}$, or $\text{Sol}_{m,n}^4 = \mathbb{R}^3 \rtimes \mathbb{R}$ ($m \neq n$), $\mathbb{R} \times \mathcal{R}$ ($m = n$). Therefore $[G, G] = \mathcal{N}$ or \mathbb{R}^3 except for $\mathbb{R} \times \mathcal{R}$. As $[G, G] = [\mathcal{R}, \mathcal{R}] = \mathbb{R}^2$ for $\mathbb{R} \times \mathcal{R}$, we obtain $\mathbb{R}^4/\Gamma = S^1 \times \mathcal{R}/\pi$.

We treat the last case that \mathbb{R} is lightlike. By an ad-hoc argument or using the result of [8], it is shown that Γ is nilpotent with $\text{Rank } C(\Gamma) = 2$. So $\mathbb{R}^4/\Gamma = S^1 \times \mathcal{N}/\Delta$ again. The universal cover $\mathbb{R} \times \mathcal{N}$ is isomorphic to the semidirect product of the translation subgroup \mathbb{R}^3 with \mathbb{R} ;

$$\mathbb{R}^3 = \left(\begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}, I \right), \quad \mathbb{R} = \left(\begin{bmatrix} \frac{-t^3}{6} \\ \frac{-t^2}{2} \\ 0 \\ t \end{bmatrix}, \begin{pmatrix} 1 & t & 0 & \frac{-t^2}{2} \\ 0 & 1 & 0 & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right).$$

Hence the lightlike action $\mathbb{R} = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}$ lies in \mathcal{N} and there is another

central group $\mathbb{R} = \begin{bmatrix} 0 \\ 0 \\ c \\ 0 \end{bmatrix}$ which constitutes a principal bundle and its

quotient:

$$\begin{aligned}\mathbb{R} &\rightarrow \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R} \times \mathbb{R}^2, \\ S^1 &\rightarrow \mathbb{R}^4/\Gamma \rightarrow S^1 \times T^2.\end{aligned}$$

As $[\Delta, \Delta] = k\mathbb{Z}$ ($\forall k \in \mathbb{Z}$), $S^1 \rightarrow \mathcal{N}/\Delta \rightarrow T^2$ is a circle bundle with euler number $k \in \mathbb{Z}$. □

Remark 3.6. *For the last case, the translation group is the same $\mathbb{R}^3 = \mathbb{R}^3 \times 0$ but \mathbb{R} has other possibilities:*

$$\begin{aligned}\mathbb{R} &= \left(\begin{bmatrix} \frac{-t^3}{6} \\ 0 \\ \frac{-t^2}{2} \\ t \end{bmatrix}, \begin{pmatrix} 1 & 0 & t & \frac{-t^2}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} \right), \\ \mathbb{R} &= \left(\begin{bmatrix} \frac{-t^3}{6} \\ \frac{-t^2}{2} \\ \frac{-t^2}{2} \\ t \end{bmatrix}, \begin{pmatrix} 1 & t & t & \frac{-t^2}{2} \\ 0 & 1 & 0 & -t \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} \right).\end{aligned}$$

4. CONFORMALLY FLAT LORENTZIAN MANIFOLD

Recall that the stabilizer of $\text{PO}(m+2, 2)$ at the point $\hat{\infty} \in S^{m+1,1}$ is isomorphic to

$$\text{PO}(m+2, 2)_{\hat{\infty}} = \mathbb{R}^{m+2} \rtimes (\text{O}(m+1, 1) \times \mathbb{R}^+) = \text{Sim}_L(\mathbb{R}^{m+2}).$$

Since a maximal amenable subgroup of $\text{O}(m+1, 1)$ is isomorphic to $\text{O}(m+1, 1)_{\infty}$ or $\text{O}(m+1, 1)_0$, a maximal amenable Lie subgroup of $\text{PO}(m+2, 2)$ is isomorphic to either one of the following groups:

$$(4.1) \quad \begin{aligned} &\text{(i)} \quad \mathbb{R}^{m+2} \rtimes (\text{Sim}(\mathbb{R}^m) \times \mathbb{Z}_2) \times \mathbb{R}^+. \\ &\text{(ii)} \quad \mathbb{R}^{m+2} \rtimes (\text{O}(m+1) \times \mathbb{Z}_2) \times \mathbb{R}^+.\end{aligned}$$

Definition 4.1. *An $m+2$ -manifold is said to be a Lorentzian parabolic manifold if it admits a $\text{Sim}(\mathbb{R}^m) \times \mathbb{R}^+$ -structure.*

As to Case (ii), we have

Proposition 4.2. *Let M be an $m+2$ -dimensional compact conformally flat Lorentzian manifold whose holonomy group belongs to $G = \mathbb{R}^{m+2} \rtimes (\text{O}(m+1) \times \mathbb{Z}_2) \times \mathbb{R}^+$. Then M is finitely covered by the Lorentz model $S^1 \times S^{m+1}$, a Hopf manifold $S^{m+1} \times S^1$, or a torus T^{m+2} .*

Proof. There exists a developing pair:

$$(4.2) \quad (P \circ \rho, P \circ \text{dev}) : (\pi_1(M), \tilde{M}) \rightarrow (\text{O}(m+2, 2)^\sim, \mathbb{R} \times S^{m+1}) \\ \rightarrow (\text{PO}(m+2, 2), S^{m+1,1}).$$

By the hypothesis, $\Gamma = P \circ \rho(\pi_1(M)) \leq G$. If Γ is a finite subgroup, it follows $\Gamma \leq \text{O}(m+1) \times \mathbb{Z}_2$ so that $P \circ \text{dev} : \tilde{M} \rightarrow S^{m+1,1}$ is a covering map. Thus $(\rho, \text{dev}) : (\pi_1(M), \tilde{M}) \rightarrow (\rho(\pi_1(M)), \mathbb{R} \times S^{m+1})$ is an equivariant diffeomorphism. There is a group extension $1 \rightarrow \mathbb{Z} \rightarrow \rho(\pi_1(M)) \rightarrow \Gamma \rightarrow 1$ associated to the covering of $\mathbb{R} \times S^{m+1}/\mathbb{Z} = S^{m+1,1}$. Then M is diffeomorphic to $\mathbb{R} \times S^{m+1}/\rho(\pi_1(M)) = S^{m+1,1}/\Gamma$.

Suppose that Γ is infinite. Recall the equivariant embedding of $(\text{Sim}_L(\mathbb{R}^{m+2}), \mathbb{R}^{m+2})$ into $(\text{PO}(m+2, 2), S^{m+1,1})$ in which \mathbb{R}^{m+2} is a dense open subset in $S^{m+1,1}$. The complement $W = S^{m+1,1} - \mathbb{R}^{m+2}$ consists of the hypersurface. (See [1].) Put $\text{Dev} = P \circ \text{dev}$ and

$$(4.3) \quad X = \tilde{M} - \text{Dev}^{-1}(W).$$

Then the developing pair reduces:

$$(4.4) \quad (\Phi, \text{Dev}) : (\pi, X) \rightarrow (\Gamma, \mathbb{R}^{m+2})$$

where $\Gamma \leq G$. Here we put $\pi = \pi_1(M)$, $\Phi = P \circ \rho$. Since $\text{O}(m+1) \times \mathbb{Z}_2 \leq \text{O}(m+2)$, X/π is endowed with the usual similarity structure.

Case 1. If X is *geodesically complete* with respect to the pull-back metric of the standard euclidean metric on \mathbb{R}^{m+2} , then Dev is a covering map of X onto \mathbb{R}^{m+2} and so Dev is a diffeomorphism. Thus Γ acts properly discontinuously on \mathbb{R}^{m+2} so that $\Gamma \leq \mathbb{R}^{m+2} \rtimes (\text{O}(m+1) \times \mathbb{Z}_2)$, i.e. there is no component in \mathbb{R}^+ . X/Γ is diffeomorphic to a euclidean space form \mathbb{R}^{m+2}/Γ . Since \mathbb{R}^{m+2} is dense in $S^{m+1,1}$, if $\tilde{M} - X \neq \emptyset$, then $\text{Dev} : \tilde{M} - X \rightarrow \text{Dev}(\tilde{M} - X)$ is a homeomorphism. Then Γ acts properly discontinuously on $\text{Dev}(\tilde{M} - X) \subset W$. Let $\Lambda = \text{Dev}(\tilde{M} - X)$. Since Λ is a Γ -invariant closed subset (and so compact), every orbit $\Gamma \cdot x$ for each $x \in \Lambda$ has an accumulation point in Λ , so Γ cannot act properly on Λ . Therefore, $\Lambda = \text{Dev}(\tilde{M} - X) = \emptyset$ or $\tilde{M} = X$. Thus M is diffeomorphic to a compact euclidean space form \mathbb{R}^{m+2}/Γ .

Hence M is finitely covered by an $m+2$ -torus $T^{m+2} = \mathbb{R}^{m+2}/\mathbb{Z}^{m+2}$.

Case 2. Suppose that a similarity manifold X is not (geodesically) complete. It follows from Fried's theorem [7] that there exists a Γ -invariant closed (affine) subspace I in \mathbb{R}^{m+2} which lies outside the developing image $\text{Dev}(X)$. (Note that a similarity manifold X/π is not necessarily compact.) In this case, some element of Γ has nontrivial \mathbb{R}^+ -summand in $G = \mathbb{R}^{m+2} \rtimes (\text{O}(m+1) \times \mathbb{Z}_2 \times \mathbb{R}^+)$. After conjugation by such element we may assume $0 \in I$.

Put the vector subspace $I = \mathbb{R}^\ell$ in \mathbb{R}^{m+2} ($\ell < m+2$). Since I is closed, the closure $\bar{\Gamma} \leq G$ leaves the complement $\mathbb{R}^{m+2} - \mathbb{R}^\ell$ invariant. This implies

$$\begin{aligned}
 \bar{\Gamma} &\leq \mathbb{R}^\ell \rtimes (\mathrm{O}(\ell) \times \mathbb{R}^+) \times \mathrm{O}(m - \ell + 1) \leq G \\
 &\parallel \\
 (4.5) \quad &\mathrm{Sim}(\mathbb{R}^\ell) \times \mathrm{O}(m - \ell + 1) \\
 &\cap \\
 &\mathrm{PO}(\ell + 1, 1) \times \mathrm{O}(m - \ell + 2).
 \end{aligned}$$

Using the real hyperbolic geometry $(\mathrm{PO}(m+3, 1), S^{m+2})$, it can be viewed as

$$\mathbb{R}^{m+2} - \mathbb{R}^\ell = S^{m+2} - S^\ell = \mathbb{H}^{\ell+1} \times S^{m-\ell+1}.$$

The subgroup of $\mathrm{PO}(m+3, 1)$ preserving this complement is isomorphic to $\mathrm{PO}(\ell+1, 1) \times \mathrm{O}(m-\ell+2)$. Thus $\mathbb{H}^{\ell+1} \times S^{m-\ell+1} = \mathbb{R}^{m+2} - \mathbb{R}^\ell$ admits a complete Riemannian metric which is invariant under this transitive group of isometries. In particular any closed subgroup acts properly on $\mathbb{R}^{m+2} - \mathbb{R}^\ell$.

Lemma 4.3 (Covering property). *X admits a π -invariant Riemannian metric such that $\mathrm{Dev} : X \rightarrow \mathbb{R}^{m+2} - \mathbb{R}^\ell$ is a covering map.*

Proof. As $\mathrm{Dev}(X)$ lies outside $I = \mathbb{R}^\ell$, it restricts the developing image $\mathrm{Dev} : X \rightarrow \mathbb{R}^{m+2} - \mathbb{R}^\ell$. Since $\bar{\Gamma}$ acts properly on $\mathbb{R}^{m+2} - \mathbb{R}^\ell$, choose a $\bar{\Gamma}$ -invariant Riemannian metric on $\mathbb{R}^{m+2} - \mathbb{R}^\ell$ such that $\mathrm{Dev} : X \rightarrow \mathbb{R}^{m+2} - \mathbb{R}^\ell$ is a local isometry with respect to the pullback metric of $\mathbb{R}^{m+2} - \mathbb{R}^\ell$. Let $P : \tilde{M} \rightarrow M$ be the covering projection. As the pullback metric on X is π -invariant, the (restricted) projection $P : X \rightarrow X/\pi$ induces a Riemannian metric on X/π .

Let $\{x_j\}$ be a Cauchy sequence in X/π . Since $X/\pi \subset M$ which is compact, $\lim_{j \rightarrow \infty} x_j = w \in M$. Choose a point $\tilde{w} \in \tilde{M}$ and neighborhoods $U(\tilde{w}) \subset \tilde{M}$, $U(w) \subset M$ such that $P : U(\tilde{w}) \rightarrow U(w)$ is a homeomorphism with $P(\tilde{w}) = w$. Let $\{\tilde{x}_j\} \subset U(\tilde{w})$ be a sequence such that $P(\tilde{x}_j) = x_j$ and $\lim_{j \rightarrow \infty} \tilde{x}_j = \tilde{w}$. As $P : U(\tilde{w}) \cap X \rightarrow U(w) \cap X/\pi$ is an isometry, $\{\tilde{x}_j\}$ is also Cauchy. Since the sequence $\{\mathrm{Dev}(\tilde{x}_j)\}$ is Cauchy in $\mathbb{R}^{m+2} - \mathbb{R}^\ell$ where $\mathbb{R}^{m+2} - \mathbb{R}^\ell$ is complete, $\lim_{j \rightarrow \infty} \mathrm{Dev}(\tilde{x}_j) = a \in \mathbb{R}^{m+2} - \mathbb{R}^\ell$.

As $\lim_{j \rightarrow \infty} \tilde{x}_j = \tilde{w}$, $\mathrm{Dev}(\tilde{w}) = a$. So $\tilde{w} \in X$ (because $\tilde{M} - X = \mathrm{Dev}^{-1}(W)$ and $a \notin W = S^{m+1,1} - \mathbb{R}^{m+2}$) and hence $P(\tilde{w}) = w \in X/\pi$, X/π is complete. So X is complete, $\mathrm{Dev} : X \rightarrow \mathbb{R}^{m+2} - \mathbb{R}^\ell$ is a covering map. \square

The proof of Lemma 4.3 works when \mathbb{R}^ℓ is replaced by the following space Y .

Proposition 4.4. *Let Y be a Γ -invariant closed subset such that the complement $\mathbb{R}^{m+2} - Y$ admits a Γ -invariant complete Riemannian metric. If $(\Psi, \text{Dev}) : (\pi, X) \rightarrow (\Gamma, \mathbb{R}^{m+2} - Y)$ is a developing pair, then $\text{Dev} : X \rightarrow \mathbb{R}^{m+2} - Y$ is a covering map.*

From Lemma 4.3, if $\ell \neq m$, $\text{Dev} : X \rightarrow \mathbb{R}^{m+2} - \mathbb{R}^\ell$ is a homeomorphism so Γ is discrete. If we recall that Γ has a nontrivial summand in \mathbb{R}^+ (**Case (2)**), (4.5) implies

$$(4.6) \quad \Gamma \leq \text{O}(\ell) \times \mathbb{R}^+ \times \text{O}(m - \ell + 1) \leq \text{O}(m + 2) \times \mathbb{R}^+.$$

If $\ell = m$, then $\text{Dev} : X \rightarrow \mathbb{R}^{m+2} - \mathbb{R}^m = \mathbb{H}_{\mathbb{R}}^{m+1} \times S^1$ is a covering map such that $\Gamma \leq \text{Sim}(\mathbb{R}^m) \times \text{O}(1)$ by Lemma 4.3. Let $p : \mathbb{H}_{\mathbb{R}}^{m+1} \times \mathbb{R}^1 \rightarrow \mathbb{H}_{\mathbb{R}}^{m+1} \times S^1$ be the projection. If $\widetilde{\text{Dev}} : X \rightarrow \mathbb{H}_{\mathbb{R}}^{m+1} \times \mathbb{R}^1$ is a lift of Dev , then it is a diffeomorphism so that the conjugate group $\tilde{\Gamma} = \widetilde{\text{Dev}} \circ \pi \circ \widetilde{\text{Dev}}^{-1}$ acts properly discontinuously on $\mathbb{H}_{\mathbb{R}}^{m+1} \times \mathbb{R}^1$. Moreover, associated with the infinite covering of $\mathbb{H}_{\mathbb{R}}^{m+1} \times S^1$, there is the commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{Sim}(\mathbb{R}^m) \times (\mathbb{Z} \rtimes \text{O}(1)) & \xrightarrow{p} & \text{Sim}(\mathbb{R}^m) \times \text{O}(1) \longrightarrow 1 \\ & & & & \uparrow & & \uparrow \\ & & & & \tilde{\Gamma} & \xrightarrow{p} & \Gamma \longrightarrow 1 \end{array}$$

Since $\tilde{\Gamma}$ is discrete and has a nontrivial summand in \mathbb{R}^+ (because so is Γ), it follows $\tilde{\Gamma} \leq \text{O}(m) \times \mathbb{R}^+ \times (\mathbb{Z} \rtimes \text{O}(1))$ which shows

$$(4.7) \quad \Gamma \leq \text{O}(m) \times \mathbb{R}^+ \times \text{O}(1) \leq \text{O}(m + 2) \times \mathbb{R}^+.$$

For both of (4.6), (4.7), Γ fixes 0 such that the complement $\mathbb{R}^{m+2} - \{0\} = S^{m+1} \times \mathbb{R}^+$ admits a complete Riemannian metric invariant under $\text{O}(m+2) \times \mathbb{R}^+$. Applying Proposition 4.4, $(\Phi, \text{Dev}) : (\pi, X) \rightarrow (\Gamma, \mathbb{R}^{m+2} - \{0\})$ is an equivariant covering map. Hence $\text{Dev} : X \rightarrow \mathbb{R}^{m+2} - \{0\}$ is a diffeomorphism. On the other hand, we can show that $\Lambda = \text{Dev}(\tilde{M} - X) = \emptyset$ as in the argument of **Case 1**, $\text{Dev} : \tilde{M} \rightarrow \mathbb{R}^{m+2} - \{0\}$ is a diffeomorphism. Hence M is finitely covered by a Hopf manifold $S^{m+1} \times S^1$. In fact, $M \cong \mathbb{R}^{m+2} - \{0\} / \Gamma = S^{m+1} \times \mathbb{R}^+ / \Gamma = S^{m+1} \times S^1 / F$ where F is a finite group of $(\text{O}(m) \times \mathbb{Z}_2) \times S^1$ acting freely. \square

Theorem 4.5. *Let M be an $m + 2$ -dimensional compact conformally flat Lorentzian manifold whose holonomy group is a virtually solvable subgroup lying in $\text{Sim}_L(\mathbb{R}^{m+2})$. Then M is either a conformally flat*

Lorentzian parabolic manifold or finitely covered by the Lorentz model $S^1 \times S^{m+1}$, a Hopf manifold $S^{m+1} \times S^1$, or a torus T^{m+2} .

Proof. Given a compact conformally flat Lorentzian $(m+2)$ -manifold M , there exists a developing pair

$$(4.8) \quad \begin{aligned} (P \circ \rho, P \circ \text{dev}) : (\pi_1(M), \tilde{M}) &\rightarrow (\text{O}(m+2, 2)^\sim, \mathbb{R} \times S^{m+1}) \\ &\rightarrow (\text{PO}(m+2, 2), S^{m+1,1}). \end{aligned}$$

Denote $\text{Aut}(T_\infty S^{m+1,1})$ the automorphism group of $T_\infty S^{m+1,1}$ where $T_\infty S^{m+1,1}$ is the tangent space of $S^{m+1,1}$ at ∞ . Let $L : \text{Sim}_L(\mathbb{R}^{m+2}) = \text{PO}(m+2, 2)_\infty \rightarrow \text{O}(m+1, 1) \times \mathbb{R}^+$ be the projection as before such that $\text{O}(m+1, 1) \times \mathbb{R}^+ \leq \text{Aut}(T_\infty S^{m+1,1})$. As $\Gamma = P \circ \rho(\pi_1(M))$ is virtually solvable in $\text{Sim}_L(\mathbb{R}^{m+2})$, there are two possibilities **(i)**, **(ii)** as in (4.1), i.e. the structure group $L(\Gamma)$ belongs to either $\text{Sim}(\mathbb{R}^m) \times \mathbb{Z}_2 \times \mathbb{R}^+$ or $\text{O}(m+1) \times \mathbb{Z}_2 \times \mathbb{R}^+$. By Definition 4.1 (cf. [15]), the case **(i)** implies that M is a conformally flat Lorentzian parabolic manifold. For the case **(ii)**, it follows $\Gamma \leq \mathbb{R}^{m+2} \rtimes (\text{O}(m+1) \times \mathbb{Z}_2) \times \mathbb{R}^+$. Hence the assertion follows from Proposition 4.2. \square

Remark 4.6. *We collect several remarks and problems.*

- (i) *If M is a compact Lorentzian similarity manifold with virtually solvable holonomy group, then it is easy to see that M is either a Lorentzian parabolic similarity manifold, a euclidean space form or a Hopf manifold.*
- (ii) *As a compact Lorentzian flat manifold is complete by Carriere's celebrated theorem [3], it is a Lorentzian parabolic similarity manifold by the definition.*
- (iii) *There is a compact incomplete Lorentzian similarity $m+2$ -manifold whose fundamental group is isomorphic to $\Gamma \times \mathbb{Z}$ where Γ is a torsionfree discrete cocompact isometry subgroup of the hyperboloid $\mathbb{H}_{\mathbb{R}}^{m+1}$. In particular, the virtual solvability of $\pi_1(M)$ does not follow from compactness for a Lorentzian similarity manifold M .*
- (iv) *Let M be a compact Lorentzian parabolic similarity manifold with virtually solvable holonomy group. Is M complete? We don't know whether there exists a compact Lorentzian parabolic similarity manifold other than compact Lorentzian flat manifolds. See Corollary 6.3 for compact Fefferman-Lorentz parabolic similarity manifold.*

For (iii), this is easily obtained by taking the interior of the cone in \mathbb{R}^{m+2} which is identified with the product $\mathbb{H}_{\mathbb{R}}^{m+1} \times \mathbb{R}^+$ on which the holonomy group $\text{O}(m+1, 1) \times \mathbb{R}^+$ acts transitively.

5. FEFFERMAN-LORENTZ PARABOLIC STRUCTURE

Let \mathbb{Z}_2 be the subgroup of the center S^1 in $U(n+1, 1)$. Put $\hat{U}(n+1, 1) = U(n+1, 1)/\mathbb{Z}_2$. The inclusion $U(n+1, 1) \rightarrow O(2n+2, 2)$ defines a natural embedding $\hat{U}(n+1, 1) \rightarrow PO(2n+2, 2)$. Then $\hat{U}(n+1, 1)$ acts transitively on $S^{2n+1,1}$ so that $(\hat{U}(n+1, 1), S^{2n+1,1})$ is a subgeometry of $(PO(2n+2, 2), S^{2n+1,1})$.

As in Introduction, a conformally flat *Fefferman-Lorentz parabolic* manifold M is a $2n+2$ -dimensional smooth manifold locally modelled on the geometry $(U(n+1, 1), S^1 \times S^{2n+1})$. See [15] for details. We observe which subgroup in $\text{Sim}_L(\mathbb{R}^{2n+2})$ corresponds to *conformally flat Fefferman-Lorentz parabolic structure*. Let $q : S^{2n+1,1} \rightarrow S^{2n+1}$ be the projection and $\{\infty\}$ the infinity point of $S^{2n+1,1}$ which maps to $\{\infty\}$ of S^{2n+1} . As a spherical *CR*-manifold, $S^{2n+1} - \{\infty\}$ is identified with the Heisenberg Lie group \mathcal{N} . Since the stabilizer is

$$PO(2n+2, 2)_{\infty} = \mathbb{R}^{2n+2} \rtimes (O(2n+1, 1) \times \mathbb{R}^+) = \text{Sim}_L(\mathbb{R}^{2n+2}),$$

the intersection $\hat{U}(n+1, 1) \cap PO(2n+2, 2)_{\infty}$ becomes

$$\hat{U}(n+1, 1)_{\infty} = \mathcal{N} \rtimes (U(n) \times \mathbb{R}^+).$$

Noting $\text{Sim}^*(\mathbb{R}^{2n}) = \mathbb{R}^{2n} \rtimes (O(2n) \times \mathbb{R}^*) \leq O(2n+1, 1)$, it follows

$$\begin{aligned} \mathcal{N} \rtimes (U(n) \times \mathbb{R}^+) &\leq \mathbb{R}^{2n+2} \rtimes (\text{Sim}^*(\mathbb{R}^{2n}) \times \mathbb{R}^+) \\ (5.1) \qquad \qquad \qquad &= (\mathbb{R}^{2n+2} \rtimes \mathbb{R}^{2n}) \rtimes (O(2n) \times \mathbb{R}^*) \times \mathbb{R}^+ \end{aligned}$$

where $\mathbb{R}^{2n+2} \rtimes \mathbb{R}^{2n}$ is a nilpotent Lie group such that $\mathcal{N} \leq \mathbb{R}^{2n+2} \rtimes \mathbb{R}^{2n}$. We have shown in [15] (Compare [6].)

Theorem 5.1. *A Fefferman-Lorentz manifold $S^1 \times N$ is conformally flat if and only if N is a spherical CR-manifold.*

Note that S^1 acts as lightlike isometries on Fefferman-Lorentz manifolds $S^1 \times N$ so does its lift \mathbb{R} on $\mathbb{R} \times N$. If $(U(n+1, 1)^{\sim}, \mathbb{R} \times S^{2n+1})$ is an infinite covering of $(\hat{U}(n+1, 1), S^{2n+1,1})$, then the subgroup $\mathbb{R} \times (\mathcal{N} \rtimes U(n))$ of $U(n+1, 1)^{\sim}$ acts transitively on the complement $\mathbb{R} \times S^{2n+1} - \mathbb{R} \cdot \infty = \mathbb{R} \times \mathcal{N}$. If $\mathbb{Z} \times \Delta$ is a discrete cocompact subgroup of $\mathbb{R} \times (\mathcal{N} \rtimes U(n))$, then we obtain (cf. [15])

Proposition 5.2. *$S^1 \times \mathcal{N}/\Delta$ is a conformally flat Lorentzian parabolic manifold on which S^1 acts as lightlike isometries.*

Remark 5.3. *In (iii) of Proposition 3.5, we saw that a finite cover of a compact (complete) Lotentzian flat 4-manifold admitting a lightlike circle S^1 is the nilmanifold $S^1 \times \mathcal{N}^3/\Delta$ with nontrivial circle bundle $S^1 \rightarrow S^1 \times \mathcal{N}^3/\Delta \rightarrow S^1 \times T^2$. The circle S^1 acts as spacelike isometries. Therefore, the 4-nilmanifold $S^1 \times \mathcal{N}^3/\Delta$ of Proposition 5.2 is not conformal to a Lorentzian flat manifold. In fact, if it admits a Lorentzian*

flat structure within the conformal class, S^1 would be spacelike as above. But S^1 is still lightlike under the conformal change of the Lorentzian metric, being contradiction.

6. DEVELOPING MAPS

Suppose that M is a $2n + 2$ -dimensional conformally flat Fefferman-Lorentz parabolic manifold. There is a developing pair:

$$(6.1) \quad (\rho, \text{dev}) : (\pi, \tilde{M}) \rightarrow (\text{U}(n+1, 1)^\sim, \tilde{S}^{2n+1,1}).$$

Let

$$(6.2) \quad \begin{aligned} q : (\text{U}(n+1, 1)^\sim, \tilde{S}^{2n+1,1}) &\rightarrow (\hat{\text{U}}(n+1, 1), S^{2n+1,1}), \\ p : (\hat{\text{U}}(n+1, 1), S^{2n+1,1}) &\rightarrow (\text{PU}(n+1, 1), S^{2n+1}) \end{aligned}$$

be the equivariant projections. Let $\Gamma = \rho(\pi)$ be the holonomy group of M as before. There is a central group extension:

$$(6.3) \quad 1 \rightarrow \mathbb{R} \rightarrow \text{U}(n+1, 1)^\sim \xrightarrow{p \circ q} \text{PU}(n+1, 1) \rightarrow 1.$$

Theorem 6.1. *Let M be a compact conformally flat Fefferman-Lorentz parabolic manifold in dimension $2n + 2$. Suppose that the holonomy group Γ is discrete. If the developing map $\text{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} = \mathbb{R} \times S^{2n+1}$ misses a closed subset which is invariant under \mathbb{R} and Γ , then dev is a covering map onto the image.*

Proof. Let Λ be both \mathbb{R} and Γ -invariant closed subset such that $\text{dev}(\tilde{M}) \subset \tilde{S}^{2n+1,1} - \Lambda$.

I. Suppose that $p \circ q(\Lambda)$ contains more than one point in S^{2n+1} . Let $\text{L}(G)$ be the *limit set* for a hyperbolic group G (cf. [5]). As $p \circ q(\Lambda)$ is invariant under $p \circ q(\Gamma)$, *Minimality of limit set* implies that $\text{L}(p \circ q(\Gamma)) \subset p \circ q(\Lambda)$. In particular, $(p \circ q)^{-1}(\text{L}(p \circ q(\Gamma))) \subset \mathbb{R} \cdot \Lambda = \Lambda$. It follows

$$(6.4) \quad \text{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - (p \circ q)^{-1}(\text{L}(p \circ q(\Gamma))).$$

(i) If $p \circ q(\Gamma)$ is discrete, then $p \circ q(\Gamma)$ acts properly discontinuously on the domain of discontinuity $S^{2n+1} - p \circ q(\Lambda)$. It is easy to see that the closure $\bar{\Gamma} \leq \text{U}(n+1, 1)^\sim$ acts properly on $\tilde{S}^{2n+1,1} - \Lambda$. Since Γ is discrete by the hypothesis, Γ acts properly discontinuously on $\tilde{S}^{2n+1,1} - \Lambda$ so there exists a Γ -invariant Riemannian metric. (Compare [16] for instance.) As $\text{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - \Lambda$ is an immersion, the pullback metric by dev is a π -invariant Riemannian metric on \tilde{M} . Thus $\text{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - \Lambda$ is a covering map.

We have a commutative diagram of group extensions from (6.3):

$$(6.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{U}(n+1, 1)^\sim & \xrightarrow{p \circ q} & \mathrm{PU}(n+1, 1) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} \cdot \Gamma & \xrightarrow{p \circ q} & p \circ q(\Gamma) \longrightarrow 1 \end{array}$$

Here $\mathbb{R} \cdot \Gamma$ is the pushout.

(ii) Suppose that $p \circ q(\Gamma)$ is not discrete, then the identity component of the closure $\overline{p \circ q(\Gamma)}^0$ is solvable by Bieberbach-Auslander's theorem [20, 8.24 Theorem]. We may assume that $\overline{p \circ q(\Gamma)}^0$ is noncompact, so it follows up to conjugacy

$$\overline{p \circ q(\Gamma)}^0 \leq \mathrm{PU}(n+1, 1)_\infty = \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+).$$

As the normalizer of $\overline{p \circ q(\Gamma)}^0$ is also contained in $\mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$, we have $p \circ q(\Gamma) \leq \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$. Hence (6.5) shows that $\Gamma \leq \mathbb{R} \cdot \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$. If we note that \mathbb{R}^+ acts as the multiplication

$$\lambda(a, z) = (\lambda^2 \cdot a, \lambda \cdot z)$$

for $\lambda \in \mathbb{R}^+$, $(a, z) \in \mathcal{N}$ (cf. [12]). Since Γ is discrete, it is easy to check

$$(6.6) \quad \begin{array}{l} \Gamma \leq \mathbb{R} \times (\mathrm{U}(n) \times \mathbb{R}^+) \text{ when } \Gamma \text{ is nontrivial in } \mathbb{R}^+, \text{ otherwise} \\ \Gamma \leq \mathbb{R} \cdot \mathcal{N} \rtimes \mathrm{U}(n). \end{array}$$

Then it follows that $\mathrm{L}(p \circ q(\Gamma)) \subset \mathrm{L}(\mathrm{U}(n) \times \mathbb{R}^+) = \{0, \infty\}$, $\mathrm{L}(p \circ q(\Gamma)) \subset \mathrm{L}(\mathcal{N} \rtimes \mathrm{U}(n)) = \{\infty\}$ respectively. Thus $(p \circ q)^{-1}(\mathrm{L}(p \circ q(\Gamma))) = \mathbb{R} \cdot \{0, \infty\}$, $(p \circ q)^{-1}(\mathrm{L}(p \circ q(\Gamma))) = \mathbb{R} \cdot \{\infty\}$ respectively. We obtain

- $\mathrm{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - \mathbb{R} \cdot \{0, \infty\} = \mathbb{R} \times (S^{2n} \times \mathbb{R}^+)$ which is a diffeomorphism. M is diffeomorphic to $\mathbb{R} \times (S^{2n} \times \mathbb{R}^+)/\Gamma$ and so M is finitely covered by $S^1 \times S^{2n} \times S^1$.
- $\mathrm{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - \mathbb{R} \cdot \{\infty\} = \mathbb{R} \times (S^{2n+1} - \{\infty\}) = \mathbb{R}^+ \times \mathcal{N}$ which is a diffeomorphism. M is diffeomorphic to $\mathbb{R} \times \mathcal{N}/\Gamma$ so that M is finitely covered by $S^1 \times \mathcal{N}/\Delta$.

In the first case, it follows $p \circ q(\Lambda) = \{0, \infty\}$. For the second case, $p \circ q(\Lambda) = \{\infty\}$ which is excluded by the assumption of Case I.

II. Suppose that $p \circ q(\Lambda)$ consists of a single point, say $\{\infty\} \in S^{2n+1}$. As $\Lambda = \mathbb{R} \cdot \infty$, it follows $\mathrm{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - \mathbb{R} \cdot \{\infty\} = \mathbb{R} \times \mathcal{N}$. Since $p \circ q(\Gamma)$ fixes $\{\infty\}$, $p \circ q(\Gamma) \leq \mathrm{PU}(n+1, 1)_\infty = \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$. As in the argument of (ii), it follows either (1) $\Gamma \leq \mathbb{R} \cdot \mathcal{N} \rtimes \mathrm{U}(n)$ or (2) $\Gamma \leq \mathbb{R} \times (\mathrm{U}(n) \times \mathbb{R}^+)$ (cf. (6.6)).

For (1), $\mathbb{R} \times \mathcal{N}$ admits an $\mathbb{R} \cdot \mathcal{N} \rtimes \mathrm{U}(n)$ -invariant Riemannian metric so $\mathrm{dev} : \tilde{M} \rightarrow \mathbb{R} \times \mathcal{N}$ is a diffeomorphism. Note that M is diffeomorphic to $\mathbb{R} \times \mathcal{N}/\Gamma$ whose finite cover $S^1 \times \mathcal{N}/\Delta$ is a conformally flat Lorentzian parabolic manifold with virtually nilpotent fundamental group.

Suppose (2) where $\Gamma \leq \mathbb{R} \times (\mathrm{U}(n) \times \mathbb{R}^+)$. As $\mathbb{R} \times (\mathrm{U}(n) \times \mathbb{R}^+)$ leaves $\mathbb{R} \times \{0\}$ invariant, put $X = \tilde{M} - \mathrm{dev}^{-1}(\mathbb{R} \times \{0\})$ which is invariant under $\mathbb{R} \times (\mathrm{U}(n) \times \mathbb{R}^+)$. This induces a developing map $\mathrm{dev} : X \rightarrow \mathbb{R} \times (\mathcal{N} - \{0\}) = \mathbb{R} \times (S^{2n} \times \mathbb{R}^+)$. Since $\mathbb{R} \times (S^{2n} \times \mathbb{R}^+)$ admits a complete Riemannian metric invariant under $\mathbb{R} \times (\mathrm{U}(n) \times \mathbb{R}^+)$, the same proof of Proposition 4.4 implies that $\mathrm{dev} : X \rightarrow \mathbb{R} \times (\mathcal{N} - \{0\})$ is a (covering) diffeomorphism. If $\mathrm{dev}^{-1}(\mathbb{R} \times \{0\}) \neq \emptyset$, then $\mathrm{dev} : \tilde{M} \rightarrow \mathrm{dev}(\tilde{M}) \subset \mathbb{R} \times \mathcal{N}$ is also a diffeomorphism. As Γ acts properly on $\mathbb{R} \times \mathcal{N}$, it follows $\mathrm{dev}(\tilde{M}) = \mathbb{R} \times \mathcal{N}$. But Γ has cohomological dimension at most 2, this cannot occur. Then $\mathrm{dev}^{-1}(\mathbb{R} \times \{0\}) = \emptyset$ which concludes that $\mathrm{dev} : \tilde{M} \rightarrow \mathbb{R} \times (\mathcal{N} - \{0\})$ is a diffeomorphism. In this case $p \circ q(\Lambda) = \{\infty\} \subset \{0, \infty\}$. This finishes the proof of the theorem. \square

Remark 6.2. According to the cases I-(i), I-(ii), II-(1) and II-(2), the following occurs:

- (a) $\mathrm{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - \Lambda$ is a covering map in which $\#p \circ q(\Lambda) \geq 2$.
- (b) $\mathrm{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - \mathbb{R} \cdot \{0, \infty\}$ is a diffeomorphism in which $p \circ q(\Lambda) = \{0, \infty\}$.
- (c) $\mathrm{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - \mathbb{R} \cdot \{\infty\}$ is a diffeomorphism in which $p \circ q(\Lambda) = \{\infty\}$.
- (d) $\mathrm{dev} : \tilde{M} \rightarrow \tilde{S}^{2n+1,1} - \mathbb{R} \cdot \{0, \infty\} = \mathbb{R} \times (\mathcal{N} - \{0\})$ is a diffeomorphism in which $p \circ q(\Lambda) = \{\infty\}$.

Corollary 6.3. There exists no $2n+2$ -dimensional compact Fefferman-Lorentz parabolic similarity manifold with discrete holonomy group.

Proof. Recall that there is an equivariant embedding of \mathbb{R}^{m+2} into $S^{m+1,1}$ with respect to $\mathrm{Sim}_L(\mathbb{R}^{m+2}) = \mathbb{R}^{m+2} \rtimes (\mathrm{O}(m+1, 1) \times \mathbb{R}^+) = \mathrm{PO}(m+2, 2)_\infty$:

$$(6.7) \quad \iota : (x, y) \rightarrow \left[\frac{|x|^2 - y^2}{2} - 1, \sqrt{2}x, \sqrt{2}y, \frac{|x|^2 - y^2}{2} + 1 \right]$$

for $x = (x_1, \dots, x_{m+1})$ and $|x| = \sqrt{x_1^2 + \dots + x_{m+1}^2}$. For $m = 2n$, let $\hat{\infty} = [1, 0, \dots, 0, 1] \in S^{2n+1,1}$. (See [12].) Then \mathbb{R}^{2n+2} misses $\hat{\infty}$ in $S^{2n+1,1}$. Moreover, the orbit of $S^1 (= \mathrm{SO}(2))$ -action at $\hat{\infty} \in S^{2n+1,1}$ becomes

$$S^1 \cdot \hat{\infty} = \{[\cos \theta, \sin \theta, 0, \dots, 0, -\sin \theta, \cos \theta], \theta \in \mathbb{R}\}.$$

In view of the formula (6.7), it follows

$$(6.8) \quad \begin{aligned} \mathbb{R}^{2n+2} &\subset S^{2n+1,1} - S^1 \cdot \infty \\ &= S^1/\mathbb{Z}_2 \times (S^{2n+1} - \{\infty\}) = S^1 \times \mathcal{N}. \end{aligned}$$

If we put $\mathcal{I} = S^1 - \{\infty\}$, then note

$$(6.9) \quad \mathbb{R}^{2n+2} = \mathcal{I} \times \mathcal{N}.$$

Putting $\Gamma = \rho(\pi)$, the developing pair reduces:

$$(6.10) \quad (\rho, \text{dev}) : (\pi, \tilde{M}) \rightarrow (\Gamma, \mathbb{R}^{2n+2}) \subset (\text{U}(n+1, 1)^\sim, \mathbb{R} \times \mathcal{N}).$$

Let $q \circ \text{dev} : \tilde{M} \rightarrow \mathbb{R}^{2n+2}$ be the developing map for which $q(\Gamma) \leq \hat{\text{U}}(n+1, 1)$. Then dev misses $\Lambda = q^{-1}(S^1 \cdot \infty)$ which is invariant under both Γ and \mathbb{R} . In particular, $p \circ q(\Lambda) = \{\infty\}$. As Γ is discrete in $\text{U}(n+1, 1)^\sim$ by the hypothesis, we can apply Theorem 6.1 to show that either (c) or (d) of Remark 6.2 occurs.

According to (c) or (d), it follows either $\Gamma \leq \mathbb{R} \times (\mathcal{N} \rtimes \text{U}(n))$ or $\Gamma \leq \mathbb{R} \times (\text{U}(n) \times \mathbb{R}^+)$. However, Γ leaves \mathbb{R}^{2n+2} invariant. As the developing image is connected, we note by (6.9) that $\text{dev}(\tilde{M}) \subset \mathcal{I} \times \mathcal{N} \subset \mathbb{R} \times \mathcal{N}$. Here \mathcal{I} is one of the components $\mathbb{Z}\mathcal{I} \subset \mathbb{R}$. This implies $\Gamma \leq \mathcal{N} \rtimes \text{U}(n)$ or $\Gamma \leq \text{U}(n) \times \mathbb{R}^+$ respectively. Then (6.10) becomes:

$$\begin{aligned} (\pi, \tilde{M}) &\xrightarrow{(\rho, \text{dev})} (\Gamma, \mathcal{I} \times \mathcal{N}) \subset (\mathcal{N} \rtimes \text{U}(n), \mathbb{R} \times \mathcal{N}), \\ (\pi, \tilde{M}) &\xrightarrow{(\rho, \text{dev})} (\Gamma, \mathcal{I} \times (\mathcal{N} - \{0\})) \subset (\text{U}(n) \times \mathbb{R}^+, \mathcal{I} \times S^{2n} \times \mathbb{R}^+). \end{aligned}$$

It follows that $M \cong \mathcal{I} \times \mathcal{N}/\Gamma$, or $M \cong \mathcal{I} \times (S^{2n} \times S^1/F)$ respectively. In each case, M cannot be compact. □

Remark 6.4. *The hypothesis that Γ is discrete is used to eliminate Case II that the limit set consists of a single point. Concerned with the hypothesis on Theorem 6.1, discreteness of the holonomy group and that Λ is \mathbb{R} -invariant may be dropped. More generally we pose*

Conjecture 6.5. *Given a compact conformally flat Lorentzian manifold, if a developing map is not surjective, then it is a covering map onto the image.*

REFERENCES

- [1] T. Barbot, V. Charette, T. Drumm, W. Goldman, K. Melnick, *A primer on the (2 + 1) Einstein universe*, 179-229, ESI Lecture notes. Math. Phys., Eur. Math. Soc., Zürich 2008.
- [2] O. Baues, *Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups*, Topology, **43**, no. 4, 903-924 (2004).
- [3] Y. Carriere, *Autour de la conjecture de L. Markus sur les varietes affines*, Invent. Math., 95, no. 3, 615-628 (1989).

- [4] V. Charette, T. A. Drumm and W. Goldman, *Affine deformations of a three-holed sphere*, Geometry & Topology 14, 1355-1382 (2010).
- [5] S.S. Chen and L. Greenberg, *Hyperbolic Spaces, Contribution to Analysis* (A Collection of Papers Dedicated to Lipman Bers, eds. L. Ahlfors and others), Academic Press, New York and London, 49-87 (1974).
- [6] C. Fefferman, *Parabolic invariant theory*, Advances in Math. 31, 131-262 (1979).
- [7] D. Fried, *Closed similarity manifolds*, Comment. Math. Helv. 55, no. 4, 576-582 (1980).
- [8] D. Fried, *Flat space times*, Jour. Differential Geometry, vol. 26, 385-396 (1987).
- [9] D. Fried and W. Goldman, *Three-dimensional affine crystallographic groups*, Adv. in Math., 47, 1-49 (1983).
- [10] W. Goldman, F. Labourie and G. Margulis, *Proper affine actions and geodesic flows of hyperbolic surfces*, Ann. of Math., 170, 1051-1083 (2009).
- [11] W. Goldman and Y. Kamishima, *The fundamental group of a compact flat Lorentz space form is virtually polycyclic*, Jour. Differential Geometry, vol. 19, 233-240, 1984.
- [12] Y. Kamishima, *Nondegenerate conformal, CR, Quaternionic CR structure on manifolds*, Handbook of Pseudo-Riemannian Geometry and Supersymmetry", (ed. Vicente Cortes), EMS in the series IRMA lectures in Mathematics and Theoretical Physics., Vol. 16, 863-896 (2010).
- [13] Y. Kamishima, *Completeness of Lorentz manifolds of constant curvature admitting Killing vector fields*, Jour. Differential Geom., vol. 37, 569-601 (1993).
- [14] Y. Kamishima, *Conformally flat manifolds whose development maps are not surjective. I*, Trans. Amer. Math. Soc., 294 607-623 (1986).
- [15] Y. Kamishima, *Conformally Lorentz parabolic structure and Fefferman Lorentz metrics*, Preprint (arXiv:0911.0867).
- [16] J. L. Koszul, *Lectures on groups of transformations*, Tata Institute of Fundamental Research, Bombay, Notes by R. R. Simha and R. Sridharan, 1965.
- [17] K.B. Lee and F. Raymond, *Seifert fiberings*, Mathematical Surveys and Monographs, vol. 166, 2010.
- [18] J. Milnor, *On fundamental groups of complete affinely flat manifolds*, Adv. Math. 25, 178-187 (1977).
- [19] G. Mess, *Lorentz spacetimes of constant curvature*, Geom. Dedicata, 126 3-45 (2007).
- [20] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Ergebnisse Math. Grenzgebiete vol. 68 Springer, Berlin, New York 1972.
- [21] C.T.C Wall, *Geometric structures on compact complex analytic surfaces*, Topology 25, 119-153 (1986).

DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, MINAMI-
OHSAWA 1-1, HACHIOJI, TOKYO 192-0397, JAPAN.

E-mail address: kami@tmu.ac.jp